Differential Geometry I $\boxed{\text{Week 2}}$

Last time: we talked about the cross product on $\mathbb{E}^3$.

Vector geometry in 2d:

Fix an orientation on $\mathbb{E}^2$ (for convenience: anti-clockwise orientation) and let $\{e_1, e_2\}$ be a positively oriented basis

- If $a = a_1e_1 + a_2e_2$ and $b = b_1e_1 + b_2e_2$ is another basis:
  It is positively oriented if $\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1b_2 - a_2b_1 > 0$

Definition:

Operator $J: \mathbb{E}^2 \to \mathbb{E}^2$ acting by rotation by $\frac{\pi}{2}$ in the positive direction:

So $J(v_1e_1 + v_2e_2) = -v_2e_1 + v_1e_2$

In matrix notation: $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (skew-symmetric)

Identifying $\mathbb{E}^2 \simeq \mathbb{C}$: J is multiplication by i.

Properties (defining J uniquely):

- $\|Jv\| = \|v\|$
- $Jv \perp v$
- if $v \neq 0$: $\{v, Jv\}$ is a positively oriented basis

Definition

Exterior product $\wedge: \mathbb{E}^2 \times \mathbb{E}^2 \to \mathbb{R}$ defined by

$$a \wedge b := \langle J(a), b\rangle = -\langle a, J(b)\rangle$$

Properties: 1) bilinear and antisymmetric

2) If $a \wedge b = 0 \Rightarrow a \parallel b$

3) $|a \wedge b| = \text{Area}(P(a,b)) = \|a\| \cdot \|b\| \cdot |\sin\theta|$

Since if $\left.\begin{array}{l} a = a_1 e_1 + a_2 e_2 \\ b = b_1 e_1 + b_2 e_2 \end{array}\right\} a \wedge b = a_1 b_2 - a_2 b_1$

$$= \det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

4) $a \wedge b > 0 \iff \{a, b\}$ is a positively oriented basis of $\mathbb{E}^2$

Definition: Oriented area:

$$\text{Area}_{or}(P(a,b)) := a \wedge b.$$

And: $\langle a, b\rangle^2 + (a \wedge b)^2 = \|a\|^2 \|b\|^2 \cos^2\theta + \|a\|^2 \|b\|^2 \sin^2\theta =$

$$= \|a\|^2 \|b\|^2$$

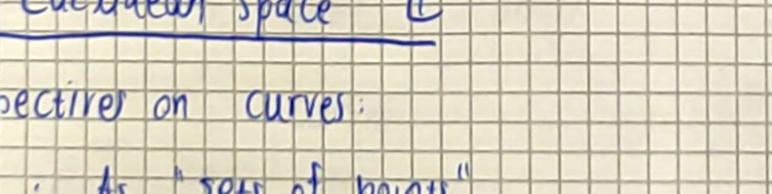
Curves in Euclidean space $\mathbb{E}^n$

Two perspectives on curves:

- Geometric: As "sets of points"

  E.g. described by equations

  $x^2 + y^2 = 1$ $\quad$ $x^2 = y^3$ $\quad$ $\begin{cases} x^2 + y^2 = 1 \\ z = 0 \end{cases}$



- Kinematic (or parametric): As trajectories of "particles"

E.g: $(x(t), y(t)) = (\cos t, \sin t)$, $(x(t), y(t)) = (t^3, t^2)$

In the latter case: "Geometric properties" are the ones who are independent of parametrization.

Basic definitions:

Fix a Cartesian coordinate system on $\mathbb{E}^n$ (i.e with respect to an orthonormal basis)

Then $\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i$, $\|v\| = \left(\sum_{i=1}^{n} v_i^2\right)^{1/2}$

↑ Not necessary; it just simplifies the formulas

(Fixing coordinates: We identify $\mathbb{E}^n$ with $\mathbb{R}^n$

and $v \longrightarrow (v_1, \dots, v_n)$

(vector) (matrix of coordinates)

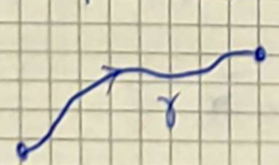
Definition:

A parametrized curve in $\mathbb{R}^n$: A continuous map $\gamma : I \longrightarrow \mathbb{R}^n$,

$\gamma : s \longrightarrow (\gamma_1(s), \dots, \gamma_n(s))$, I: interval in $\mathbb{R}$.

(interval of parametrization)

- $s \in I$: parameter



- The set $\gamma(I) = \{\gamma(s) \mid s \in I\} \subseteq \mathbb{R}^n$:

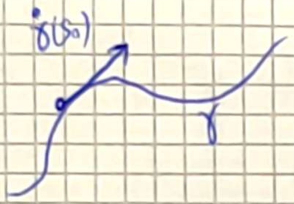
Trace of the ~~curve~~ parametrized curve

(or simply curve)

Definition: γ is *differentiable* at $s_0 \in I$ if

$$\frac{d\gamma}{ds}(s_0) = \lim_{s \to s_0} \frac{\gamma(s) - \gamma(s_0)}{s - s_0} \quad \text{exists.}$$

The limit: *Velocity vector*, $\dot{\gamma}(s_0)$ or $\gamma'(s_0)$.



Speed: $V_\gamma(s_0) := \|\dot{\gamma}(s_0)\|$.

Lemma: If $\gamma : I \to \mathbb{R}^n$ is differentiable at $s_0 \in I$, then

$$\dot{\gamma}(s_0) = \left(\frac{d\gamma_1}{ds}(s_0), \dots, \frac{d\gamma_n}{ds}(s_0) \right) = (\dot{\gamma}_1(s_0), \dots, \dot{\gamma}_n(s_0))$$

and $$V_\gamma(s_0) = \sqrt{\left(\frac{d\gamma_1}{ds}\right)^2(s_0) + \dots + \left(\frac{d\gamma_n}{ds}(s_0)\right)^2}$$

(if the basis is orthonormal)

Proof: Easy ☒

Def: • γ is of class C^k if the coordinate functions $\gamma_j : I \to \mathbb{R}$ are of class C^k, $j = 1, \dots, n$,

i.e. $\forall j = 1, \dots, n$: $\forall\ 0 \leq m \leq k$, $\frac{d^m \gamma_j}{ds^m}$ exists and is continuous on I.

• C^∞: it is of class C^k for all $k \geq 0$.

• γ is *piecewise* C^k if it is C^0 and there exists a finite partition of $I = [a,b)$

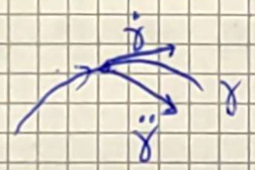
into $[a,s_0] \cup [s_0,s_1] \cup \dots \cup [s_m, b)$

such that on each subinterval: γ is of class C^k.

so at the partition points: γ is continuous, but the derivatives might have jump discontinuities (e.g.: curve with corners)

A few more definitions:

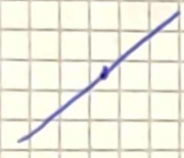
- If γ is of class C^2: <u>acceleration</u> $\ddot{\gamma} = \frac{d^2\gamma}{ds^2}$



- A curve $\gamma : I \to \mathbb{R}^n$ is
  - <u>regular at s_0</u> if it's differentiable there and $\dot{\gamma}(s_0) \neq 0$ (otherwise: <u>singular</u>)
  - <u>regular</u> if it is C^1 and $\dot{\gamma} \neq 0$ everywhere on I
  - <u>biregular at s_0</u> if it is twice differentiable there and $\{\dot{\gamma}(s_0), \ddot{\gamma}(s_0)\}$ are linearly independent
  - <u>biregular</u> if it is C^2 and biregular at every point.

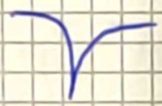
__Note:__ The question of regularity of a (parametrized) curve depends on the parametrization:

If $\gamma(t) = (t^3, t^3)$ then $\dot{\gamma}(0) = 0$



But $\tilde{\gamma}(s) = (s,s)$ has the same trace, but is regular.

However: For $\alpha(s) = (s^2, s^3)$



there is no reparametrization that makes it regular at the cusp.

If γ is __regular__ at s_0 but __not__ __biregular__: Then any reparametrization of γ will have this property (later Exercise)

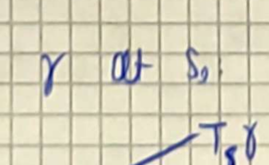
- If $\gamma : I \to \mathbb{R}^n$ and J is a subinterval of I:

  $\gamma|_J$ is an __arc__ of the curve

- If $s_1, s_2 \in I$, $s_1 \neq s_2$ but $\gamma(s_1) = \gamma(s_2)$: Double point

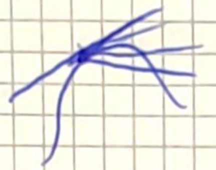
- __Simple__ curve: 1-1 (no double points).

__Def:__ • Tangent line of γ at s_0: $T_{s_0}\gamma : \lambda \to \gamma(s_0) + \lambda \cdot \dot{\gamma}(s_0)$, $\lambda \in \mathbb{R}$



(defined if γ __regular__ at s_0)

Tangent: The limit of chords $[\gamma(s_0), \gamma(s_n)]$ as $s_n \to s_0$



- <u>Osculating plane</u> (for $\gamma: I \to \mathbb{R}^n$, $n \geq 3$):

If γ is biregular at s_0, this is the plane spanned by $\{\dot{\gamma}(s_0), \ddot{\gamma}(s_0)\}$

- It contains the tangent line
- It is the limiting plane defined by <u>three points</u> on the curve as they converge to $\gamma(s_0)$

<u>Examples</u>

1) Cubic curve in $\mathbb{R}^3$: $\gamma(s) = (as, bs^2, cs^3)$, $a, b, c \neq 0$.

C^∞ curve, $\dot{\gamma} = (a, 2bs, 3cs^2) \neq (0,0,0)$

$\ddot{\gamma} = (0, 2b, 6cs)$

$\therefore \dot{\gamma} \times \ddot{\gamma} \neq 0$

So <u>biregular</u>

Speed: $V_\gamma(s) = \|\dot{\gamma}\| = \sqrt{a^2 + 4b^2s^2 + 9c^2s^4}$

At $s = 0$: $\gamma(0) = \vec{0}$

$\dot{\gamma}(0) = (a, 0, 0)$

$\ddot{\gamma}(0) = (0, 2b, 0)$

So: Tangent line: $T_0\gamma = \{e_1\}$

Oscullating plane: $\Pi_0\gamma = \{e_1, e_2\}$

2) The curve $\beta: \mathbb{R} \to \mathbb{R}^n$,

$\beta(u) = (u^2, u^3, \dots, u^{n+1})$.

$\dot{\beta}(u) = (2u, 3u^2, \dots, (n+1)u^n)$, unique singular point: at $u = 0$ $(\beta(0) = \vec{0})$

3) Line passing through $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$:

If $w = p - q$

then $\gamma(t) = q + t \cdot w, \quad t \in \mathbb{R}$

$\|\dot{\gamma}\| = \|w\| = \text{const} \neq 0$ (so regular)

But $\ddot{\gamma} = 0$ (so not biregular)

4) If $\gamma(t) = p + t^3 w$: Different parametrization of same line

(but this is now singular at $t = 0$)

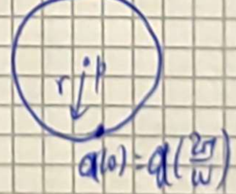
5) Circle centered at $p \in \mathbb{R}^n$, in the 2-plane $\Pi \subseteq \mathbb{R}^n$ of radius $r > 0$:

If $\{e_1, e_2\}$ is an orthonormal basis of the plane Π translated to contain the origin, then:

$$\alpha(t) = p + r \cdot \left(\cos(\omega t) \cdot e_1 + \sin(\omega t) \cdot e_2 \right),$$

$$0 \leq t \leq \frac{2\pi}{\omega}, \quad \omega > 0 \text{ any given constant.}$$

- C^∞ curve
- $\alpha(0) = \alpha\left(\frac{2\pi}{\omega}\right)$ (double point)



$$\dot{\alpha}(t) = r \cdot \omega \left(-\sin(\omega t) e_1 + \cos(\omega t) \cdot e_2 \right)$$

So $V_\alpha(t) = \|\dot{\alpha}\| = r \cdot \omega \neq 0$ (regular) (constant speed)

$$\ddot{\alpha}(t) = r \cdot \omega \cdot \left(-\cos(\omega t) e_1 - \sin(\omega t) \cdot e_2 \right) \Longrightarrow$$

It is biregular ($\ddot{\alpha} \perp \dot{\alpha}$ and both non-zero)

Exercise: If γ has constant speed, then $\ddot{\gamma} \perp \dot{\gamma}$

6) Graph of a function:

If $f: I \to \mathbb{R}$ is differentiable,

then $\gamma_f : I \to \mathbb{R}^2$, $\gamma(x) = (x, f(x))$ parametrizes the graph

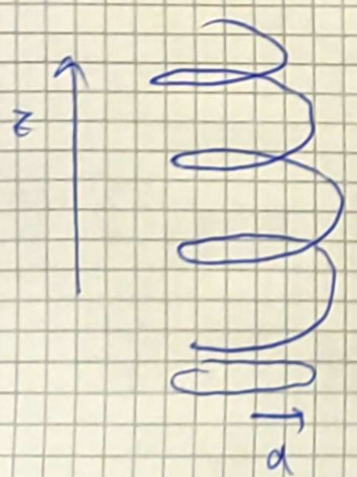
and $\dot{\gamma}_f(x) = (1, f'(x))$

So $V_{\gamma_f}(s) = \|\dot{\gamma}_f(s)\| = \sqrt{1 + (f'(s))^2}$ (always regular)

8) Circular helix:

$h: \mathbb{R} \to \mathbb{R}^3$, $h(t) = (a \cdot \cos t, a \cdot \sin t, b \cdot t)$

where $a, b \neq 0$



$\dot{h}(t) = (-a \cdot \sin t, a \cdot \cos t, b)$

$\ddot{h}(t) = (-a \cos t, -a \sin t, 0)$

So biregular,

$\|\dot{h}\| = \sqrt{a^2 + b^2}$ constant

Length of a curve

Definition: Let $\gamma : [a,b] \to \mathbb{R}^n$ be a C^1 curve.

Length $\ell(\gamma) = \int_a^b \|\dot{\gamma}\| \, dt$

• Makes sense for piecewise C^1 curves.

Examples

1) If γ has constant speed $\|\dot{\gamma}\| = v = \text{const.}$

$$\ell(\gamma) = v \cdot (b-a)$$

For line segment from p to q:
$\gamma(t) = p + t \cdot (q-p)$, $0 \leq t \leq 1$
$\ell(\gamma) = \|p-q\|$

2) Circle in $\mathbb{R}^2$ with center at p and radius r:

$$\gamma(\theta) = p + r \cdot \cos\theta \, e_1 + r \cdot \sin\theta \, e_2$$

$$= (p_1 + r \cdot \cos\theta, \; p_2 + r \cdot \sin\theta)$$

$$\dot{\gamma} = (-r \cdot \sin\theta, \; r \cdot \cos\theta) \Rightarrow \|\dot{\gamma}\| = r = \text{const}$$

So: For arc from θ_1 to θ_2:

$$\ell(\gamma|_{[\theta_1,\theta_2]}) = r \cdot (\theta_2 - \theta_1)$$

3) If γ is the graph of $f : [a,b] \to \mathbb{R}$

$$\gamma(x) = (x, f(x)) \quad , \quad \|\dot{\gamma}\| = \sqrt{1 + (f')^2}$$

So $\ell(\gamma) = \int_a^b \sqrt{1+(f')^2} \, dx$

Proposition: If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a similarity transformation of scale $\lambda > 0$,

$\gamma: [a,b] \to \mathbb{R}^n$ is a C^1 curve,

$\tilde{\gamma}: [a,b] \to \mathbb{R}^n$ is $\tilde{\gamma}(t) = T(\gamma(t))$

then $\ell(\tilde{\gamma}) = \lambda \cdot \ell(\gamma)$

Proof:

Since T is a similarity transformation of scale λ, it takes the form

$$T(x) = \lambda \cdot Ax + b, \quad \text{where } A \in O(n)$$

So

$$\tilde{\gamma}(t) = T(\gamma(t)) = \lambda \cdot A\gamma(t) + b$$

$$\Rightarrow \frac{d}{dt}\tilde{\gamma}(t) = \lambda \cdot A \cdot \frac{d\gamma}{dt} \xrightarrow{\|Ax\| = \|x\|} \left\| \frac{d\tilde{\gamma}}{dt} \right\| = \lambda \left\| \frac{d\gamma}{dt} \right\|$$

$$\text{So } \ell(\tilde{\gamma}) = \int_a^b \left\| \frac{d\tilde{\gamma}}{dt} \right\| dt = \lambda \cdot \ell(\gamma) \quad \blacksquare$$

Proposition (additivity of the length):

If $\alpha: [a,b] \to \mathbb{R}^n$ is a C^1 curve and $c \in [a,b]$,

let $\beta: [a,c] \to \mathbb{R}^n$ be the restriction of α on $[a,c]$

and $\gamma: [c,b] \to \mathbb{R}^n$ 〃 〃 〃 〃 $[c,b]$

then $\ell(\alpha) = \ell(\beta) + \ell(\gamma)$